Incompressible Maxwell-Boussinesq approximation: Existence, uniqueness and shape sensitivity

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Abstract

We prove the existence and uniqueness of weak solutions to the variational formulation of the Maxwell-Boussinesq approximation problem. Some further regularity in $W^{1,2+\delta}$, $\delta > 0$, is obtained for the weak solutions. The shape sensitivity analysis by the boundary variations technique is performed for the weak solutions. As a result, the existence of the strong material derivatives for the weak solutions of the problem is shown. The result can be used to establish the shape differentiability for a broad class of shape functionals for the models of Fourier-Navier-Stokes flows under the electromagnetic field.

Keywords. magnetohydrodynamic flows, shape sensivity.

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1 Introduction

The problem of magnetohydrodynamics flows have been studied by several authors [9, 11, 16, 17, 19, 20, 21, 26], and it goes back to the work of Ladyzhenskaya and Solonnikov. At that time the coupled system did not include thermal effects. The full complete problem including the heat transfer seems to be more realistic and not many authors were dealt with it. The full Navier-Stokes-Fourier-Maxwell problem was only partially studied in the works [1, 20, 21], where the principal coefficients are assumed constant. While in [9] the coefficients are only temperature dependent, and the force term is either globally bounded (truncated) or that the thermal expansion coefficient is sufficiently small. Concerning the shape sensitivity analysis we can mention work of Zolésio and his collaborators (see [2, 3, 8, 12, 13, 14, 22, 23, 24, 25]), where the case of Navier-Stokes problem was investigated and also the coupled problem with heat transfer. We refer to [7] where an uncoupled complete problem is studied (cf. Remark 6.1). It is only in this paper that the problem under study has the principal coefficients varying with the temperature as well as the space variable.

Let Ω be an open bounded subset of $\mathbb{R}^3$ with the boundary $\partial\Omega \in C^{1,1}$ which is splitted into two parts $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$, where Γ_D is an open nonempty subset of $\partial\Omega$ and $\Gamma_N = \partial\Omega \setminus \bar{\Gamma}_D$. The thermoelectromagnetoflow problem reads in Ω:

$$-\nabla \cdot (\nu(T)D\mathbf{u}) + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mu \mathrm{rot}\mathbf{H} \times \mathbf{H} + \rho\mathbf{f} - \rho\mathbf{G}(T)T; \tag{1}$$

$$\nabla \times (\sigma^{-1}(T)\nabla \times \mathbf{H}) = \nabla \times (\sigma^{-1}(T)\mathbf{J}_0 + \mu\mathbf{u}\times\mathbf{H}); \tag{2}$$

$$\mathrm{div}\mathbf{u} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = \mathrm{div}\mathbf{H} = 0; \tag{3}$$

$$-\nabla \cdot (k(T)\nabla T) + \rho\mathbf{u}\cdot\nabla T = f. \tag{4}$$

Here $\mathbf{u}$ is the fluid velocity vector, T is the temperature, $D\mathbf{u} = (D_{ij}) = (\partial_i u_j + \partial_j u_i)/2$ $(i,\ j = 1,2,3)$ is the symmetrized gradient of the velocity, μ the magnetic permeability, p denotes the pressure, ρ is density, $\mathbf{f}$ and f denote the external forces and heat sources, respectively. The coefficients ν, σ, k are temperature dependent functions representing the viscosity of the fluid, the electric and the heat conductivities, respectively. Indeed in order to be more realistic setting these coefficients are not only functions on the temperature but also on the space variable. The density ρ is assumed to be constant, we set $\rho = 1$.

The buoyancy force as in the Boussinesq approximation is described by $\mathbf{G}(T) = \beta(T)(0,0,g)^\top$, where β denotes the coefficient of thermal dilatation and g is the constant of gravity. The existence of two body forces in the fluid, the Lorentz force $\mathbf{J} \times \mathbf{B} = (\nabla \times \mathbf{H}) \times (\mu \mathbf{H})$ and the buoyancy force, results from the presence of the magnetic field $\mathbf{H}$. Moreover (2) results if we take the rotational in the second equation of the steady-state Maxwell equations:

$$\nabla \times \mathbf{E} = 0; \qquad \mathbf{J} = \nabla \times \mathbf{H},$$

where $\mathbf{E}$ is the electric intensity field and $\mathbf{J}$ is the current density given by the Ohm's law

$$\mathbf{J} = \mathbf{J}_0 + \sigma(T)(\mathbf{E} + \mathbf{u} \times \mathbf{B}),$$

where $\mathbf{J}_0$ denotes a given applied current [1].

Finally, the thermoelectromagnetoflow problem under study has the following boundary conditions

$$\mathbf{u} = \mathbf{0}, \qquad \mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega; \tag{5}$$

$$T = 0 \text{ on } \Gamma_D, \quad k(T)\frac{\partial T}{\partial \mathbf{n}} + \alpha T = h \quad \text{on} \quad \Gamma_N, \tag{6}$$

where α represents the convective heat tranfer coefficient. Here α is a function only depending on the space variable. We refer to [4, 5, 9, 21] where it can be extended to a function also depending on the temperature and even to include radiation effects. To simplify the presentation it is assumed a homogeneous Dirichlet condition for the velocity of the fluid (cf. Remark 3.5).

The outline of the paper is as follows. In Section 2, new existence results are stated under diverse assumptions for the system of strongly coupled elliptic equations governing temperature dependent electromagnetic stationary flow. Fluid velocity, magnetic field intensity and fluid temperature are the state variables. Section 3 is devoted to the proof of the existence of a weak solution to the nonlinear coupling of electromagnetics, heat and fluid device, using a fixed point argument. Some well posedness auxiliary existence results are established as well as results on the regularity of solutions. In Section 4, additional regularity of a weak solution to the considered electromagnetic flow problem is proved, assuming more regular external forces and applied current intensities. Assuming Lipschitz type continuity of function parameters with respect to temperature, this solution is shown to be unique for suitable small data. In Section 5, assuming that all coefficients of the elliptic system are constant and the velocity field is divergence free, sensitivity analysis of the unique solution to the considered elliptic system with respect to perturbation of the boundary of the domain occupied by the fluid is performed using the material derivative approach. The existence of strong material derivative of the weak solution to the elliptic system is shown. The elliptic system characterizing this derivative is provided.

2 Assumptions and main existence results

We need some assumptions on the model, which are listed below.

Let us assume that

(H1) $\nu, \sigma, k : \Omega \times \mathbb{R} \to \mathbb{R}$ are Caratheodory functions such that

$$\exists \nu^{\#}, \nu_{\#} > 0 : \quad \nu_{\#} \leq \nu(\cdot, \xi) \leq \nu^{\#}, \text{ a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}; \tag{7}$$

$$\exists \sigma^{\#}, \sigma_{\#} > 0 : \quad \sigma_{\#} \leq \sigma(\cdot, \xi) \leq \sigma^{\#}, \text{ a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}; \tag{8}$$

$$\exists k^{\#}, k_{\#} > 0 : \quad k_{\#} \leq k(\cdot, \xi) \leq k^{\#}, \text{ a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}; \tag{9}$$

(H2) $\mathbf{G} = (0, 0, G)$ where $G = g\beta$ with β a real, continuous, and bounded function and we denote by $G^{\#}$ the upper bound for the function G;

(H3) $\alpha \in L^q_+(\Gamma_N) = \{\alpha \in L^q(\Gamma_N) : \quad \alpha \geq 0\}$ for q such that $q > 3/2$, which means that its conjugate $q' = q/(q-1)$ verifies $q' < 3$;

(H4) and $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $\quad \mathbf{J}_0 \in \mathbf{L}^2(\Omega)$, $\quad f \in L^2(\Omega) \quad$ and $h \in L^2(\Gamma_N)$.

In the framework of function spaces of the Lebesgue and Sobolev type, the norms are denoted by the symbols $\|\cdot\|, \|\cdot\|_1, \|\cdot\|_{\Gamma_N}$ in spaces $L^2(\Omega), H^1(\Omega)$, $L^2(\Gamma_N)$, respectively, and there scalar and vector function spaces are not distinguished in our notations. Providing that the meaning remains clear, the canonical norm in $L^p(\Omega)$ for $p \neq 1, 2$ is denoted by $\|\cdot\|_p$. We introduce the Hilbert spaces

$$\begin{aligned}
\mathbf{V} &= \{\mathbf{v} \in \mathbf{H}^1_0(\Omega) : \quad \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega\} \,, \\
\mathbf{V}(rot) &= \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \ \operatorname{rot} \mathbf{v} \in \mathbf{L}^2(\Omega), \ \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\} \,, \\
Z &= \{\xi \in H^1(\Omega) : \quad \xi = 0 \text{ on } \Gamma_D\} \,,
\end{aligned}$$

equipped with their standard scalar products. We recall that the norms $\|\cdot\|_{\mathbf{V}(rot)}$ and $\|\cdot\|_Z$ are equivalent to the usual seminorms $\|\nabla \times \cdot\|$ and $\|\nabla \cdot\|$

and also to the norms $\|\cdot\|_1$ on spaces $\mathbf{H}^1(\Omega)$ and $H^1(\Omega)$, respectively (cf. [11]).

We state the main results of the paper.

Theorem 2.1. *Under the above assumptions (H1)-(H4), and, in addition, under the following assumptions*

$$
\begin{aligned}
b > 0 \quad \textit{and} \quad & \mu a^2 < b^3 , \\
& a = \frac{\nu_\#}{\mu\sigma_\#}\|\mathbf{J}_0\| , \\
& b = \frac{\nu_\#}{\mu\sigma^\#} - \left(\|\mathbf{f}\| + \frac{G^\#}{k_\#}(\|f\| + \|h\|_{\Gamma_N})\right),
\end{aligned}
\tag{10}
$$

the problem (1)-(6) has a weak solution in the following sense:

The triplet $(\mathbf{u}, \mathbf{H}, T) \in \mathbf{V} \times \mathbf{V}(rot) \times Z$ satisfies the following integral identities

$$
\begin{aligned}
&\int_\Omega \nu(T) D\mathbf{u} : D\mathbf{v} dx + \int_\Omega (\mathbf{v} \otimes \mathbf{u}) : \nabla \mathbf{u} dx = \\
&= \int_\Omega \Big(\mu(\nabla \times \mathbf{H}) \times \mathbf{H} + \mathbf{f} - \mathbf{G}(T)T\Big) \cdot \mathbf{v} dx, \quad \forall \mathbf{v} \in \mathbf{V};
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&\int_\Omega \frac{1}{\sigma(T)}(\nabla \times \mathbf{H}) \cdot (\nabla \times \mathbf{v}) dx = \mu \int_\Omega (\mathbf{u} \times \mathbf{H}) \cdot (\nabla \times \mathbf{v}) dx + \\
&+ \int_\Omega \frac{1}{\sigma(T)} \mathbf{J}_0 \cdot (\nabla \times \mathbf{v}) dx, \quad \forall \mathbf{v} \in \mathbf{V}(rot);
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&\int_\Omega k(T) \nabla T \cdot \nabla \eta dx + \int_\Omega \mathbf{u} \cdot \nabla T \eta dx + \int_{\Gamma_N} \alpha T \eta ds = \\
&= \int_\Omega f \eta dx + \int_{\Gamma_N} h \eta ds, \quad \forall \eta \in Z.
\end{aligned}
\tag{13}
$$

Moreover, the pair $(\mathbf{H}, T)$ enjoys the additional regularity, actually belongs to $\mathbf{W}^{1,2+\epsilon}(\Omega) \times W^{1,2+\varepsilon}(\Omega)$ for some ϵ, $\varepsilon > 0$.

Remark 2.2. If $\epsilon > 2/5$ we can deduce the additional regularity on $\mathbf{u}$ as in [7]. Otherwise, since the operators in the above elliptic equations of the second order have discontinuous coefficients, we can obtain Hölder continuity on $\bar{\Omega}$ of the weak solution T due to the De Giorgi-Nash Theorem if $f, h \in L^q(\Omega)$ for $q > 3$. If σ is taken as a continuous function, then the main operator in (12) has continuous coefficient and the regularity theory can be applied to the weak solution $\mathbf{H}$. Or simply if we suppose that the electric conductivity σ is constant, it will be sufficient to our purposes in the study of the shape sensivity. However, in the sequel the data assumptions are kept as general as possible.

Theorem 2.3. *Let $\epsilon_0 < \epsilon < 1$ and $2 < q < 3$ be such that*

$$\frac{3q}{3-q} = \frac{(2+\epsilon_0)(2+\epsilon)}{\epsilon - \epsilon_0}. \tag{14}$$

If $\mathbf{J}_0 \in \mathbf{L}^q(\Omega)$, then $\mathbf{H} \in \mathbf{L}^{3q/(3-q)}(\Omega)$. Under the assumption $\mathbf{f} \in \mathbf{L}^{2+\delta_1}(\Omega)$, where $\delta_1 > 0$, the weak solution $\mathbf{u}$ given by Theorem 2.1 enjoys the additional regularity, actually belongs to $\mathbf{W}^{1,2+\delta}(\Omega)$ for some $\delta > 0$. Furthermore, under the following Lipschitz-type continuity assumption on the temperature dependent function parameters of the model

$$\exists \bar{\nu} > 0 : \qquad |\nu(T^2) - \nu(T^1)| \leq \bar{\nu}|T^2 - T^1|^{3\delta/(2+\delta)}, \tag{15}$$
$$\exists \bar{G} > 0 : \qquad g|\beta(T^2) - \beta(T^1)| \leq \bar{G}|T^2 - T^1|, \tag{16}$$
$$\exists \bar{\sigma} > 0 : \qquad |\sigma(T^2) - \sigma(T^1)| \leq \bar{\sigma}|T^2 - T^1|^{3\epsilon/(2+\epsilon)}, \tag{17}$$
$$\exists \bar{k} > 0 : \qquad |k(T^2) - k(T^1)| \leq \bar{k}|T^2 - T^1|^{3\varepsilon/(2+\varepsilon)}, \quad \forall T^2, T^1 \in \mathbb{R}, \tag{18}$$

the weak solution $(\mathbf{u}, \mathbf{H}, T)$ is unique for small data.

The existence of the pressure p in the space of distributions follows from the well-known results by using the divergence-free test functions $\mathbf{v} \in \mathbf{C}_0^\infty(\Omega)$ in (11). Moreover, the pressure is unique up to a constant.

3 Proof of Theorem 2.1

First, we recall the Tychonoff extension to weak topologies of the Schauder fixed point theorem [10, pp. 453-456 and 470].

Theorem 3.1. *Let K be a nonempty weakly sequentially compact convex subset of a locally convex linear topological vector space V. Let $\mathcal{L} : K \to K$ be a weakly sequentially continuous operator. Then $\mathcal{L}$ has at least one fixed point.*

Let $\mathcal{L}$ be the mapping of the form

$$\mathcal{L} : (\mathbf{w}, \mathbf{h}, \xi) \in \mathbf{V} \times \mathbf{V}(rot) \times Z \mapsto (\mathbf{H}, T) \mapsto (\mathbf{u}, \mathbf{H}, T) \ ,$$

where the functions $\mathbf{u}$, $\mathbf{H}$ and T are the solutions for the elliptic boundary value problems (23), (21) and (19), respectively. Indeed, the fixed point argument starts by prescribing $(\mathbf{w}, \mathbf{h}, \xi)$ from $\mathbf{V} \times \mathbf{V}(rot) \times Z$ next by finding the temperature and the magnetic field and finally the velocity of the fluid. The proofs of such existence results are the straightforward application of the classical existence theory, hence are omitted here.

Proposition 3.2. *Let $(\mathbf{w}, \xi) \in \mathbf{V} \times Z$ and assume that conditions (9), (H3)-(H4) are fulfilled. Then there exists a unique $T \in Z$ such that*

$$\int_\Omega k(\xi)\nabla T \cdot \nabla \eta dx + \int_\Omega \mathbf{w} \cdot \nabla T \eta dx + \int_{\Gamma_N} \alpha T \eta ds = \\ = \int_\Omega f \eta dx + \int_{\Gamma_N} h \eta ds, \quad \forall \eta \in Z. \tag{19}$$

Moreover, the energy estimate holds

$$k_\# \|T\|_1 \leq \|f\| + \|h\|_{\Gamma_N}. \tag{20}$$

Proposition 3.3. *Let $(\mathbf{w}, \mathbf{h}, \xi) \in \mathbf{V} \times \mathbf{V}(rot) \times Z$ and assume that conditions (8) and (H4) are fulfilled. Then there exists a unique $\mathbf{H} \in \mathbf{V}(rot)$ such that*

$$\int_\Omega \frac{1}{\sigma(\xi)} (\nabla \times \mathbf{H}) \cdot (\nabla \times \mathbf{v}) dx = -\mu \int_\Omega (\mathbf{h} \times \mathbf{w}) \cdot (\nabla \times \mathbf{v}) dx + \\ + \int_\Omega \frac{1}{\sigma(\xi)} \mathbf{J}_0 \cdot (\nabla \times \mathbf{v}) dx, \quad \forall \mathbf{v} \in \mathbf{V}(rot). \tag{21}$$

Moreover, the energy estimate holds

$$\frac{1}{\sigma^\#} \|\mathbf{H}\|_1 \leq \mu \|\mathbf{h} \times \mathbf{w}\| + \frac{1}{\sigma_\#} \|\mathbf{J}_0\|. \tag{22}$$

Proposition 3.4. *Let $(\mathbf{w}, \xi) \in \mathbf{V} \times Z$, T and $\mathbf{H}$ the solutions in accordance with Propositions 3.2 and 3.3, respectively, and assume that conditions (7), (H2) and (H4) are fulfilled. Then there exists a unique $\mathbf{u} \in \mathbf{V}$ such that*

$$\int_\Omega \nu(\xi) D\mathbf{u} : D\mathbf{v} dx + \int_\Omega (\mathbf{v} \otimes \mathbf{w}) : \nabla \mathbf{u} dx = \\ = \int_\Omega \Big(\mu (\nabla \times \mathbf{H}) \times \mathbf{H} + \mathbf{f} - \mathbf{G}(T) T \Big) \cdot \mathbf{v} dx, \quad \forall \mathbf{v} \in \mathbf{V}. \tag{23}$$

Moreover, the energy estimate holds

$$\nu_\# \|\mathbf{u}\|_1 \leq \mu \|\nabla \times \mathbf{H}\| \|\mathbf{H}\|_{L^3} + \|\mathbf{f}\| + G^\# \|T\|_{L^{6/5}}. \tag{24}$$

Remark 3.5. For given $\mathbf{g} \in \mathbf{H}^{-1/2}(\partial\Omega)$, there exists a lifting $\mathbf{u_g} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{u_g} = \mathbf{g}$ on $\partial\Omega$ and $\mathbf{u_g}$ verifies

$$-\nabla \cdot (\nu(\xi) D\mathbf{u_g}) + (\mathbf{w} \cdot \nabla)\mathbf{u_g} = -\nabla p_{\mathbf{g}}; \quad \nabla \cdot \mathbf{u_g} = 0 \text{ in } \Omega.$$

If the element $\mathbf{U} = \mathbf{u} - \mathbf{u_g} \in \mathbf{V}$ is determined by a solution to the problem

$$-\nabla \cdot (\nu(\xi) D\mathbf{U}) + (\mathbf{w} \cdot \nabla)\mathbf{U} = -\nabla p_{\mathbf{U}} + \mu(\nabla \times \mathbf{H}) \times \mathbf{H} + \mathbf{f} - \mathbf{G}(T)T \text{ in } \Omega,$$

then $\mathbf{u} = \mathbf{U} + \mathbf{u_g}$ is the solution to the problem

$$\begin{aligned} -\nabla \cdot (\nu(\xi) D\mathbf{u}) + (\mathbf{w} \cdot \nabla)\mathbf{u} = -\nabla p + \mu(\nabla \times \mathbf{H}) \times \mathbf{H} + \mathbf{f} - \mathbf{G}(T)T \text{ in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega, \quad \mathbf{u} = \mathbf{g} \text{ on } \partial\Omega. \end{aligned}$$

Therefore, it is assumed that $\mathbf{g} = \mathbf{0}$, observing that in the inhomogeneous case a smallness assumption for the velocity at the boundary will be also needed in order to prove the main results.

In view of Propositions 3.2, 3.3 and 3.4, the operator $\mathcal{L}$ is well defined. Moreover, $\mathcal{L}$ maps the ball

$$\begin{aligned} K = \{(\mathbf{w}, \mathbf{h}, \xi) \in \mathbf{V} \times \mathbf{V}(rot) \times Z : \quad & \|\mathbf{w}\|_1 \leq R_1,\ \|\mathbf{h}\|_1 \leq R_2, \\ & \|\xi\|_1 \leq \frac{1}{k_\#}(\|f\| + \|h\|_{\Gamma_N})\} \end{aligned}$$

into itself, since by (20), (22) and (24) it follows that

$$\|\mathbf{H}\|_1 \leq \sigma^\# \left(\mu R_1 R_2 + \frac{1}{\sigma_\#}\|\mathbf{J}_0\| \right) \leq R_2 \ , \tag{25}$$

$$\|\mathbf{u}\|_1 \leq \frac{1}{\nu_\#} \left(\mu R_2^2 + \|\mathbf{f}\| + \frac{G^\#}{k_\#}(\|f\| + \|h\|_{\Gamma_N}) \right) = R_1 \ , \tag{26}$$

where $R_2 > 0$ is such that

$$\frac{\mu\sigma^\#}{\nu_\#} R_2 \left(\mu R_2^2 + \|\mathbf{f}\| + \frac{G^\#}{k_\#}(\|f\| + \|h\|_{\Gamma_N}) \right) + \frac{\sigma^\#}{\sigma_\#}\|\mathbf{J}_0\| \leq R_2$$

or equivalently

$$a \leq R_2(b - \mu R_2^2)$$

if $b > 0$ and $(a/b)^2 < b/\mu$ which is assured by (10).

In order to apply Theorem 3.1 it remains to prove the weak continuity of $\mathcal{L}$. Since we have the compact embeddings

$$\mathbf{V}, \mathbf{V}(rot) \hookrightarrow\hookrightarrow \{\mathbf{w} \in \mathbf{L}^4(\Omega) : \quad \nabla \cdot \mathbf{w} = 0 \text{ in } \Omega, \ \mathbf{w} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\} \tag{27}$$

$$Z \hookrightarrow\hookrightarrow L^1(\Omega), \tag{28}$$

let $\{(\mathbf{w}_m, \mathbf{h}_m, \xi_m)\}$ be a sequence such that

$$\mathbf{w}_m \to \mathbf{w}, \quad \mathbf{h}_m \to \mathbf{h} \quad \text{in } \mathbf{L}^4(\Omega); \qquad \xi_m \to \xi \text{ in } L^1(\Omega). \tag{29}$$

Let $(\mathbf{u}_m, \mathbf{H}_m, T_m)$ be the corresponding weak solutions given by Propositions 3.2, 3.3 and 3.4, for each $m \in \mathbb{N}$. From the estimates (24), (22) and (20), the sequence $\{(\mathbf{u}_m, \mathbf{H}_m, T_m)\}$ is bounded in $\mathbf{V} \times \mathbf{V}(rot) \times Z$. Then there exists the weak limit $(\mathbf{u}, \mathbf{H}, T) \in \mathbf{V} \times \mathbf{V}(rot) \times Z$ such that

$$\mathbf{u}_m \rightharpoonup \mathbf{u} \text{ in } \mathbf{V}; \qquad \mathbf{H}_m \rightharpoonup \mathbf{H} \text{ in } \mathbf{V}(rot); \qquad T_m \rightharpoonup T \text{ in } Z, \tag{30}$$

possibly for a subsequence, still denoted by $(\mathbf{u}_m, \mathbf{H}_m, T_m)$. Applying (27)-(28) we obtain

$$\mathbf{u}_m \to \mathbf{u}, \quad \mathbf{H}_m \to \mathbf{H} \quad \text{in } \mathbf{L}^4(\Omega); \qquad T_m \to T \text{ in } L^1(\Omega). \tag{31}$$

We pass to the limit as $m \to +\infty$ in the integral identities (23), (21) and (19), in which replacing $\mathbf{w}, \mathbf{h}, \xi, \mathbf{u}, \mathbf{H}$ and T by the sequences $\mathbf{w}_m$, $\mathbf{h}_m$, ξ_m, $\mathbf{u}_m$, $\mathbf{H}_m$ and T_m, respectively, using(29)-(31) and the continuity properties of the Niemytskii operators in the coefficients combined with (7)-(9). Therefore, we conclude that the limit $(\mathbf{u}, \mathbf{H}, T)$ is a solution corresponding to $(\mathbf{w}, \mathbf{h}, \xi)$ of the required problem (23), (21) and (19).

Then Theorem 3.1 guarantees the existence of at least one fixed point which is the required weak solution.

The regularity $(\mathbf{H}, T) \in \mathbf{W}^{1,2+\epsilon}(\Omega) \times W^{1,2+\varepsilon}(\Omega)$ for some ϵ, $\varepsilon > 0$ is a consequence of the following regularity results.

Proposition 3.6. *If $\mathbf{J}_0 \in \mathbf{L}^2(\Omega)$ then there exists a constant $\epsilon > 0$ such that the weak solution $\mathbf{H} \in \mathbf{V}(rot)$ of (12) belongs to $\mathbf{W}^{1,2+\epsilon}(\Omega)$, i.e.*

$$\|\nabla \mathbf{H}\|_{2+\epsilon} \leq K_1,$$

with a constant $K_1 > 0$ only dependent on the data.

Proof. Adapting the regularity theory for elliptic equations of the second order [18], we obtain $\mathbf{H} \in \mathbf{W}^{1,2+\epsilon}(\Omega)$ with $2+\epsilon < 6$ since

$$\mathbf{J}_0 - \sigma(T)\mu\mathbf{H} \times \mathbf{u} \in \mathbf{L}^2(\Omega) \hookrightarrow (\mathbf{W}^{1,6/5}(\Omega))'.$$

The following result is consequence of the regularity of solutions to the mixed boundary value problems for elliptic equations (cf. [18]).

Proposition 3.7. *If $f \in L^2(\Omega)$ and $h \in L^2(\Gamma_N)$ then there exists a constant $\varepsilon > 0$ such that the weak solution $T \in Z$ of (13) belongs to $W^{1,2+\varepsilon}(\Omega)$, i.e.*

$$\|\nabla T\|_{2+\varepsilon} \leq K_2,$$

with a constant $K_2 > 0$ only dependent on the data.

Proof. According to [18] we obtain $T \in W^{1,2+\varepsilon}(\Omega)$ with $2+\varepsilon < 3$ since $f, h \in (W^{1,3/2}(\Omega))'$.

4 Proof of Theorem 2.3

The regularity $\mathbf{u} \in \mathbf{W}^{1,2+\delta}(\Omega)$ for some $\delta > 0$ is a consequence of the following regularity results.

Proposition 4.1. *For every $2 < q < 3$, if $\mathbf{J}_0 \in \mathbf{L}^q(\Omega)$ then the weak solution $\mathbf{H} \in \mathbf{V}(rot)$ of (12) belongs to $\mathbf{L}^{3q/(3-q)}(\Omega)$.*

Proof. Adapting the regularity theory for elliptic equations of the second order [18], the desired result is obtained provided by

$$\mathbf{J}_0 - \sigma(T)\mu\mathbf{H} \times \mathbf{u} \in \mathbf{L}^q(\Omega).$$

Proposition 4.2. *If q is given as in (14) and $\mathbf{f} \in \mathbf{L}^{2+\delta_1}(\Omega)$ for some $\delta_1 > 0$, then there exists a constant $\delta > 0$ such that the weak solution $\mathbf{u} \in \mathbf{V}$ of (11) belongs to $\mathbf{W}^{1,2+\delta}(\Omega)$, i.e.*

$$\|\nabla \mathbf{u}\|_{2+\delta} \leq K_3,$$

with a constant $K_3 > 0$ only dependent on the data.

Proof. For every $x_0 \in \bar{\Omega}$, $0 < r < R$ small enough, $\Omega(x_0, R) := \Omega \cap B(x_0, R)$, $\theta \in]0,1[$ and some positive constants B_1, B_2, independent of $\mathbf{u}, \mathbf{H}$ and T, we have the following reverse estimate (cf. [6, Lemma 3.2])

$$\begin{aligned}
\left(\int_{\Omega(x_0,r)} |\nabla \mathbf{u}|^2 dx\right)^{1/2} \quad \leq \quad & \theta \left(\int_{\Omega(x_0,R)} |\nabla \mathbf{u}|^2 dx\right)^{1/2} \\
& + \frac{B_1}{R-r} \left(\int_{\Omega(x_0,R)} |\nabla \mathbf{u}|^{6/5} dx\right)^{5/6} \\
& + \frac{B_2}{R-r} \left(\int_{\Omega(x_0,R)} (|\mathbf{u} \otimes \mathbf{u}|^2 + |\mathbf{F}|^2 + |\mathbf{f}|^2 + 1) dx\right)^{1/2}
\end{aligned}$$

where $\mathbf{F} = \mu \mathrm{rot}\mathbf{H} \times \mathbf{H} - \mathbf{G}(T)T$. By Propositions 3.6 and 4.1, we have $\mathbf{H} \in \mathbf{W}^{1,2+\epsilon}(\Omega) \cap \mathbf{L}^{(2+\epsilon_0)(2+\epsilon)/(\epsilon-\epsilon_0)}(\Omega)$. Thus it follows that $\mathrm{rot}\mathbf{H} \times \mathbf{H} \in \mathbf{L}^{2+\epsilon_0}(\Omega)$ and $\mathbf{F} \in \mathbf{L}^{2+\epsilon_0}(\Omega)$. Since $\mathbf{u} \otimes \mathbf{u} \in \mathbf{L}^3(\Omega)$ then the Gehring inequality [15] guarantees the higher integrability $\mathbf{u} \in \mathbf{W}^{1,2+\delta}(\Omega)$ for some $0 < \delta < \min\{\epsilon_0, \delta_1\}$.

Now, we prove the uniqueness. To this end, let $(\mathbf{u}^1, \mathbf{H}^1, T^1)$ and $(\mathbf{u}^2, \mathbf{H}^2, T^2)$ be two weak solutions to problem (11), (13), and (12). Arguing as in [7], the respective differences $\bar{\mathbf{u}} = \mathbf{u}^1 - \mathbf{u}^2$, $\bar{\mathbf{H}} = \mathbf{H}^1 - \mathbf{H}^2$ and $\bar{T} = T^1 - T^2$ satisfy

$$\begin{aligned}
\frac{\nu_\#}{2} \|D\bar{\mathbf{u}}\|^2 \leq \frac{\bar{\nu}}{\nu_\#} \|\bar{T}\|_6^{6\delta/(2+\delta)} \|D\mathbf{u}^2\|_{2+\delta}^2 + C_2^2 \|D\bar{\mathbf{u}}\|^2 \|\nabla \mathbf{u}^2\| + \\
+ \frac{C_1}{\nu_\#} \Big(\mu \|(\nabla \times \mathbf{H}^1) \times \mathbf{H}^1 - (\nabla \times \mathbf{H}^2) \times \mathbf{H}^2\|_{6/5} + \\
+ G^\# \|\bar{T}\|_{6/5} + \bar{G} \|\bar{T}\|_6 \|T^2\|_{3/2} \Big)^2; \\
\frac{1}{4\sigma^\#} \|\nabla \times \bar{\mathbf{H}}\|^2 \leq \sigma^\# \left\| \left(\frac{1}{\sigma(T^2)} - \frac{1}{\sigma(T^1)} \right) \nabla \times \mathbf{H}^2 \right\|^2 + \\
+ \sigma^\# \mu \|\mathbf{u}^1 \times \mathbf{H}^1 - \mathbf{u}^2 \times \mathbf{H}^2\|^2 + \sigma^\# \left\| \left(\frac{1}{\sigma(T^1)} - \frac{1}{\sigma(T^2)} \right) \mathbf{J}_0 \right\|^2; \\
\frac{k_\#}{2} \|\nabla \bar{T}\|^2 \leq \frac{\bar{k}}{k_\#} \|\bar{T}\|_6^{6\varepsilon/(2+\varepsilon)} \|\nabla T^2\|_{2+\varepsilon}^2 + \frac{C_1}{k_\#} \|\bar{\mathbf{u}}\|_6^2 \|\nabla T^2\|_{3/2}^2, \qquad (32)
\end{aligned}$$

where C_1, C_2 are the Sobolev constants of the embeddings $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and $H^1(\Omega) \hookrightarrow L^4(\Omega)$, respectively. Using the Lipschitz continuity assumptions (15)-(18), and applying Hölder and Young inequalities leads to

$$\frac{\nu_\#}{2}\|D\bar{\mathbf{u}}\|^2 \leq \frac{\bar{\nu}}{\nu_\#}\|\bar{T}\|_6^{6\delta/(2+\delta)}\|D\mathbf{u}^2\|_{2+\delta}^2 + C_2^2\|D\bar{\mathbf{u}}\|^2\|\nabla\mathbf{u}^2\| +$$
$$+\frac{C_1}{\nu_\#}\Big(\mu\|\nabla\times\bar{\mathbf{H}}\|\|\mathbf{H}^1\|_3 + \mu\|\nabla\times\mathbf{H}^2\|\|\bar{\mathbf{H}}\|_3 + G^\#\|\bar{T}\|_{6/5} + \bar{G}\|\bar{T}\|_6\|T^2\|_{3/2}\Big)^2;$$
$$\frac{1}{4(\sigma^\#)^2}\|\nabla\times\bar{\mathbf{H}}\|^2 \leq \frac{\bar{\sigma}}{(\sigma_\#)^2}\|\bar{T}\|_6^{6\epsilon/(2+\epsilon)}(\|\nabla\mathbf{H}^2\|_{2+\epsilon}^2 + \|\mathbf{J}_0\|_{2+\epsilon}^2) +$$
$$+\mu(\|\bar{\mathbf{u}}\|_4^2\|\mathbf{H}^1\|_4^2 + \|\mathbf{u}^2\|_4^2\|\bar{\mathbf{H}}\|_4^2).$$

Let K_1, K_2 and K_3 be the upper bounds derived in Propositions 3.6, 3.7 and 4.2, respectively, and K_4 stand for the upper bound in estimate (20), namely,

$$K_4 = \frac{1}{k_\#}(\|f\| + \|h\|_{\Gamma_N}).$$

Next, in view of (25)-(26), we set

$$R_1 = \frac{1}{\nu_\#}\left(\mu R_2^2 + \|\mathbf{f}\| + G^\# K_4\right),$$

where R_2 is chosen such that

$$\Big(1 - \frac{\mu\sigma^\#}{\nu_\#}\left(\mu R_2^2 + \|\mathbf{f}\| + G^\# K_4\right)\Big)R_2 = \frac{\sigma^\#}{\sigma_\#}\|\mathbf{J}_0\|,$$

we have

$$\frac{\nu_\#}{2}\|D\bar{\mathbf{u}}\|^2 \leq \frac{\bar{\nu}}{\nu_\#}\|\bar{T}\|_6^{6\delta/(2+\delta)}K_3^2 + C_2^2\|D\bar{\mathbf{u}}\|^2 R_1 +$$
$$+\frac{C_1}{\nu_\#}\Big(\mu R_2(\|\nabla\times\bar{\mathbf{H}}\| + \|\bar{\mathbf{H}}\|_3) + G^\#\|\bar{T}\|_{6/5} + \bar{G}\|\bar{T}\|_6 K_4\Big)^2;$$
$$\frac{1}{4(\sigma^\#)^2}\|\nabla\times\bar{\mathbf{H}}\|^2 \leq \frac{\bar{\sigma}}{(\sigma_\#)^2}\|\bar{T}\|_6^{6\epsilon/(2+\epsilon)}(K_1^2 + \|\mathbf{J}_0\|_{2+\epsilon}^2) +$$
$$+\mu(\|\bar{\mathbf{u}}\|_4^2 R_2^2 + R_1^2\|\bar{\mathbf{H}}\|_4^2).$$

Now, sum the above two inequalities with (32) rewritten as follows as

$$\frac{k_\#}{2}\|\nabla\bar{T}\|^2 \leq \frac{\bar{k}}{k_\#}\|\bar{T}\|_6^{6\varepsilon/(2+\varepsilon)}K_2^2 + \frac{C_1}{k_\#}\|\bar{\mathbf{u}}\|_6^2 K_4^2.$$

As a result,

$$
\begin{aligned}
&\left(\frac{\nu_{\#}}{2} - C^2 R_1 - C\mu R_2^2 - \frac{C}{k_{\#}} K_4^2\right) \|D\bar{\mathbf{u}}\|^2 + \\
&+ \left(\frac{1}{4(\sigma^{\#})^2} - \frac{2C\mu^2}{\nu_{\#}} R_2^2 - C\mu R_1^2\right) \|\nabla \bar{\mathbf{H}}\|^2 + \\
&+ \left(\frac{k_{\#}}{2} - \frac{C}{\nu_{\#}} (\bar{\nu} K_3^2 + (G^{\#} + \bar{G} K_4)^2) - \right. \\
&\left. - \frac{C\bar{\sigma}}{(\sigma_{\#})^2} (K_1^2 + \|\mathbf{J}_0\|_{2+\epsilon}^2) - \frac{C\bar{k}}{k_{\#}} K_2^2\right) \|\nabla \bar{T}\|^2 \leq 0,
\end{aligned}
$$

with C standing for different Sobolev constants, and the uniqueness of solution holds under smallness assumption on the data.

5 Shape sensivity analysis

In this section we deal with the shape sensivity analysis to the model correspondent to Theorem 2.1, when the coefficients ν, k, σ and α are assumed constants. First, a family of mappings $\mathcal{T}_\tau : \mathbb{R}^3 \to \mathbb{R}^3$ associated with a given velocity field $V(\tau, x)$ is constructed. The evolution of geometrical domains, if the vector field V is chosen, is governed by the real parameter τ. So we denote by $\Omega_\tau = \mathcal{T}_\tau(\Omega)$ the variable domain depending on *two parameters*, a vector field V and the real variable τ. We call the family of perturbations Ω_τ of a given initial configuration Ω, and the variable τ has the meaning of the time in our setting. Indeed the variable domains Ω_τ are defined by the images of the mapping which is given by the system of differential equations

$$
\frac{d}{d\tau} x(\tau) = V(\tau, x(\tau)), \ x(0) = X,
$$

with the solution denoted by $x(\tau) = x(\tau, X), \tau \in (0, \tau_1), \ X \in \mathbb{R}^3$, for some positive constant τ_1. Therefore each specific family parametrized by τ is defined in the direction of a given vector field V, and it is denoted by $\Omega_\tau = \{x \in \mathbb{R}^3 | \ x = x(\tau, X), X \in \Omega\}$.

In our setting all equations defined in variable domain Ω_τ can be transported to the reference domain which is also called the fixed domain Ω, using the inverse transformation $\mathcal{T}_\tau^{-1} : \Omega_\tau \to \Omega$.

Let us assume the following additional hypothesis:

(H5) The field V is compactly supported with respect to the spatial variable x, i.e.,

$$V \in C(0, \tau_1; \mathcal{D}^2(\Omega; \mathbb{R}^3)), \text{ supp}V \subset \Omega,$$

and it is divergence free.

(H6) In the variable domain setting, the elements

$$\mathbf{f}^\tau \in \mathbf{L}^2(\Omega_\tau), \quad \mathbf{J}_0^\tau \in \mathbf{L}^2(\Omega_\tau), \quad f^\tau \in L^2(\Omega_\tau) \quad \text{and } h^\tau \in L^2(\Gamma_N^\tau), \tag{33}$$

stand for the data in boundary value problems in Ω_τ, are simply given by restrictions to Ω_τ of some functions

$$\mathbf{f} \in \mathbf{H}^1(\mathbb{R}^3), \quad \mathbf{J}_0 \in \mathbf{H}^1(\mathbb{R}^3), \quad f \in H^1(\mathbb{R}^3) \quad \text{and } h \in H^1(\mathbb{R}^3) \tag{34}$$

defined in the whole space. In this way the shape derivatives of all the data vanish, except for h, and the material derivatives are just given by the scalar products of the gradients of the data with respect to spatial variables with the velocity vector field, e.g., $\dot{f} = \nabla f \cdot V$, provided that all data are given in the Sobolev spaces $H^1(\mathbb{R}^3)$.

Notice that (H5) implies the additional constraint, $|\Omega| = \text{constant}$, this means that for our shapes the admissible domains have the given volume.

5.1 Perturbated problem

We consider in (H5) that the velocity field $V(\tau, x)$ is divergence free, which implies that also our $\mathbf{u}$ and $\mathbf{H}$ also conserve the divergenceless. This simplifies the situation and we do not need to apply Bogovskii operator, since for pressure we use the standard Rham theorem.

Definition 5.1. We say a perturbated problem to the model (1)-(6) in a perturbated domain to the following system of equations in Ω_τ

$$-\nabla \cdot (\nu D\mathbf{u}^\tau) + (\mathbf{u}^\tau \cdot \nabla)\mathbf{u}^\tau + \nabla p^\tau = \mu \text{rot}\mathbf{H}^\tau \times \mathbf{H}^\tau + \mathbf{f}^\tau - \mathbf{G}(T^\tau)T^\tau; \tag{35}$$

$$\nabla \times (\nabla \times \mathbf{H}^\tau) = \nabla \times (\mathbf{J}_0^\tau + \sigma\mu\mathbf{u}^\tau \times \mathbf{H}^\tau); \tag{36}$$

$$\text{div}\mathbf{u}^\tau = \text{div}\mathbf{H}^\tau = 0; \tag{37}$$

$$-\nabla \cdot (k\nabla T^\tau) + \mathbf{u}^\tau \cdot \nabla T^\tau = f^\tau; \tag{38}$$

with the boundary conditions:

$$\mathbf{u}^\tau = \mathbf{0}, \qquad \mathbf{H}^\tau \cdot \mathbf{n}^\tau = 0 \quad \text{on } \partial\Omega_\tau; \tag{39}$$

$$T^\tau = 0 \quad \text{on } \Gamma_D^\tau; \quad k\frac{\partial T^\tau}{\partial \mathbf{n}^\tau} + \alpha T^\tau = h^\tau \quad \text{on } \Gamma_N^\tau. \tag{40}$$

We introduce the Hilbert spaces

$$\begin{aligned}
\mathbf{V}^\tau &= \{\mathbf{v} \in \mathbf{H}_0^1(\Omega_\tau) : \quad \text{div } \mathbf{v} = 0 \text{ in } \Omega_\tau\} \\
\mathbf{V}^\tau(rot) &= \{\mathbf{v} \in \mathbf{L}^2(\Omega_\tau) : \text{ rot } \mathbf{v} \in \mathbf{L}^2(\Omega_\tau), \\
&\qquad \text{div } \mathbf{v} = 0 \text{ in } \Omega_\tau, \ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_\tau\} \\
Z^\tau &= \{\xi \in H^1(\Omega_\tau) : \quad \xi = 0 \text{ on } \Gamma_D^\tau\}
\end{aligned}$$

equipped with their standard inner products.

Theorem 5.2. *Assuming (H2), (H5), (33) and (10) with the constants a and b under the perturbated data, i.e.*

$$\begin{aligned}
a &= \frac{\nu}{\mu\sigma}\|\mathbf{J}_0^\tau\| \\
b &= \frac{\nu}{\mu\sigma} - \left(\|\mathbf{f}^\tau\| + \frac{G\#}{k}(\|f^\tau\| + \|h^\tau\|_{\Gamma_N})\right),
\end{aligned}$$

then the problem (35)-(40) has a weak solution in the following sense:

The triple $(\mathbf{u}^\tau, \mathbf{H}^\tau, T^\tau) \in \mathbf{V}^\tau \times \mathbf{V}^\tau(rot) \times Z^\tau$ *and it satisfies*

$$\nu \int_{\Omega_\tau} D\mathbf{u}^\tau : D\mathbf{v}^\tau dx_\tau + \int_{\Omega_\tau} D\mathbf{u}^\tau : (\mathbf{u}^\tau \otimes \mathbf{v}^\tau) dx_\tau =$$

$$= \int_{\Omega_\tau} \Big(\mu(\nabla \times \mathbf{H}^\tau) \times \mathbf{H}^\tau + \mathbf{f}^\tau - \mathbf{G}(T^\tau)T^\tau\Big) \cdot \mathbf{v}^\tau dx_\tau, \quad \forall \mathbf{v}^\tau \in \mathbf{V}^\tau;$$

$$\int_{\Omega_\tau} (\nabla \times \mathbf{H}^\tau) \cdot (\nabla \times \mathbf{w}^\tau) dx_\tau = \sigma\mu \int_{\Omega_\tau} (\mathbf{u}^\tau \times \mathbf{H}^\tau) \cdot (\nabla \times \mathbf{w}^\tau) dx_\tau +$$

$$+ \int_{\Omega_\tau} \mathbf{J}_0^\tau \cdot (\nabla \times \mathbf{w}^\tau) dx_\tau, \quad \forall \mathbf{w}^\tau \in \mathbf{V}^\tau(rot);$$

$$k \int_{\Omega_\tau} \nabla T^\tau \cdot \nabla \eta^\tau dx_\tau + \int_{\Omega_\tau} \mathbf{u}^\tau \cdot \nabla T^\tau \eta^\tau dx_\tau + \alpha \int_{\Gamma_N^\tau} T^\tau \eta^\tau ds_\tau =$$

$$= \int_{\Omega_\tau} f^\tau \eta^\tau dx_\tau + \int_{\Gamma_N^\tau} h^\tau \eta^\tau ds_\tau, \quad \forall \eta^\tau \in Z^\tau.$$

Proof. See the proof of Theorem 2.1.

Theorem 5.3. *If the assumptions of Theorem 5.2 are fulfilled, the solution* $(\mathbf{u}^\tau, \mathbf{H}^\tau, T^\tau)$ *in accordance to Theorem 5.2 is such that* $(\mathbf{H}^\tau, T^\tau)$ *belongs to* $\mathbf{W}^{1,2+\epsilon}(\Omega_\tau) \times W^{1,2+\varepsilon}(\Omega_\tau)$ *for some* ϵ, $\varepsilon > 0$. *Moreover, if we assume* $\mathbf{f}^\tau \in$

$\mathbf{L}^{2+\delta_1}(\Omega_\tau)$ *for some* $\delta_1 > 0$ *and* $\mathbf{J}_0^\tau \in \mathbf{L}^q(\Omega_\tau)$ *with* q *given as in (14), then* $\mathbf{u}^\tau \in \mathbf{W}^{1,2+\delta}(\Omega_\tau)$ *for some* $\delta > 0$*, and* $(\mathbf{u}^\tau, \mathbf{H}^\tau, T^\tau)$ *is unique under small data.*

Proof. See the proof of Theorems 2.1 and 2.3.

5.2 Transported problem

The transported solution to the fixed domain is denoted by $\mathbf{u}_\tau = \mathbf{u}^\tau \circ \mathcal{T}_\tau$, $\mathbf{H}_\tau = \mathbf{H}^\tau \circ \mathcal{T}_\tau$, $T_\tau = T^\tau \circ \mathcal{T}_\tau$ with data $\mathbf{f}_\tau = \mathbf{f}^\tau \circ \mathcal{T}_\tau$, $\mathbf{G}_\tau = \mathbf{G}^\tau \circ \mathcal{T}_\tau$, $\mathbf{J}_{0\tau} = \mathbf{J}_0^\tau \circ \mathcal{T}_\tau$, $f_\tau = f^\tau \circ \mathcal{T}_\tau$ and $h_\tau = h^\tau \circ \mathcal{T}_\tau$.

We begin by recalling the result [27].

Proposition 5.4. *The unit normal vector field on* Γ_τ *is given by*

$$\mathbf{n}_\tau(\mathcal{T}_\tau(X)) = (\|{}^*J\mathcal{T}_\tau^{-1} \cdot \mathbf{n}\|_{\mathbb{R}^3}^{-1} {}^*(D\mathcal{T}_\tau)^{-1} \cdot \mathbf{n})(X)$$

for $X \in \Gamma$*. Here we denote by* $J\mathcal{T}_\tau$ *the Jacobian of* $\mathcal{T}_\tau$ *and for any matrix* B *the transposed matrix is denoted by* ${}^*B$*. For any* $f \in L^1(\Gamma_\tau)$*,*

$$\int_{\Gamma_\tau} f ds_\tau = \int_\Gamma f \circ \mathcal{T}_\tau \|M(\mathcal{T}_\tau) \cdot \mathbf{n}\|_{\mathbb{R}^3} ds,$$

where $M(\mathcal{T}_\tau) = det\ (J\mathcal{T}_\tau)^* J\mathcal{T}_\tau^{-1}$ *is the cofactor matrix of the Jacobian matrix* $J\mathcal{T}_\tau$*.*

We recall the following important results, which give us the answer on the question, what happens with grad, div or curl after applying the transformation of domain.

Proposition 5.5. *Denote by* $J\mathcal{T}_\tau$ *the Jacobian of* $\mathcal{T}_\tau$ *and for any matrix* B *the transposed matrix is denoted by* ${}^*B$*. Then we have*

(i) $(grad\ w) \circ \mathcal{T}_\tau = \left({}^*J\mathcal{T}_\tau^{-1}\nabla\right)\left(w \circ \mathcal{T}_\tau\right)$ *for all* $w \in H^1(\Omega)$*;*

(ii) $(div\ \mathbf{w}) \circ \mathcal{T}_\tau = \zeta(\tau)^{-1}\left(\zeta(\tau) J\mathcal{T}_\tau^{-1}\nabla\right) \cdot \left(\mathbf{w} \circ \mathcal{T}_\tau\right)$ *for all* $\mathbf{w} \in \mathbf{H}^1(\Omega)$*;*

(iii) $(curl\ \mathbf{w}) \circ \mathcal{T}_\tau = \left({}^*J\mathcal{T}_\tau^{-1}\nabla\right) \times \left(\mathbf{w} \circ \mathcal{T}_\tau\right)$ *for all* $\mathbf{w} \in \mathbf{H}^1(\Omega)$*.*

Remark 5.6. From Proposition 5.5, it follows that functions which are divergence free on Ω_τ generally loose this property when they are transported to the fixed domain. For more details see [19, Remark 6.2]. This is a reason, why we assume additional for simplicity that div $V = 0$.

We introduce the following notations

$$\begin{aligned}
\zeta(\tau) &= \det(J\mathcal{T}_\tau), \\
\varrho(\tau) &= {}^*J\mathcal{T}_\tau^{-1}, \\
A(\tau) &= \zeta(\tau)\, {}^*\varrho(\tau)\varrho(\tau), \\
B(\tau) &= \zeta(\tau)\varrho(\tau), \\
\omega(\tau) &= \|M(J\mathcal{T}_\tau)\cdot \mathbf{n}\|_{\mathbb{R}^3}.
\end{aligned}$$

Definition 5.7. We call the transported problem to the following system of equations

$$\nu \int_\Omega A(\tau) : (D\mathbf{u}_\tau D\mathbf{v}_\tau) dx + \int_\Omega B(\tau)\nabla \mathbf{u}_\tau : (\mathbf{v}_\tau \otimes \mathbf{u}_\tau) dx =$$

$$= \int_\Omega \zeta(\tau)\Big(\mu((\varrho(\tau)\nabla) \times \mathbf{H}_\tau) \times \mathbf{H}_\tau + \mathbf{f}_\tau - \mathbf{G}(T_\tau)T_\tau\Big) \cdot \mathbf{v}_\tau dx, \quad \forall \mathbf{v}_\tau \in \mathbf{V};$$

$$\int_\Omega ((\varrho(\tau)\nabla) \times \mathbf{H}_\tau) \cdot ((\varrho(\tau)\nabla) \times \mathbf{w}_\tau) =$$

$$= \int_\Omega \zeta(\tau)(\sigma\mu \mathbf{u}_\tau \times \mathbf{H}_\tau + \mathbf{J}_{0\tau}) \cdot ((\varrho(\tau)\nabla) \times \mathbf{w}_\tau) dx, \quad \forall \mathbf{w}_\tau \in \mathbf{V}(rot);$$

$$k \int_\Omega A(\tau) : (\nabla T_\tau \otimes \nabla \eta_\tau) dx + \int_\Omega B(\tau) : (\mathbf{u}_\tau \otimes \nabla T_\tau)\eta_\tau dx +$$

$$+\alpha \int_{\Gamma_N} T_\tau \eta_\tau \omega(\tau) ds = \int_\Omega f_\tau \eta_\tau \zeta(\tau) dx + \int_{\Gamma_N} h_\tau \eta_\tau \omega(\tau) ds, \quad \forall \eta_\tau \in Z.$$

Theorem 5.8. *Suppose that the assumptions (H2), (H5) and (34) are fulfilled and additionally assuming that (10) holds for the constants a and b under the transported data:*

$$\begin{aligned}
a &= \frac{\nu}{\mu\sigma}\|\mathbf{J}_{0\tau}\| \\
b &= \frac{\nu}{\mu\sigma} - \left(\|\mathbf{f}_\tau\| + \frac{G^\#}{k}(\|f_\tau\| + \|h_\tau\|_{\Gamma_N})\right),
\end{aligned}$$

then the triple $(\mathbf{u}_\tau, \mathbf{H}_\tau, T_\tau) \in \mathbf{V} \times \mathbf{V}(rot) \times Z$ *is a weak solution in the sense of Definition 5.7. Moreover, the solution* $(\mathbf{u}_\tau, \mathbf{H}_\tau, T_\tau)$ *is such that* $(\mathbf{H}_\tau, T_\tau)$

belongs to $\mathbf{W}^{1,2+\epsilon}(\Omega)\times W^{1,2+\varepsilon}(\Omega)$ *for some* $\epsilon,\,\varepsilon>0$*, and if* $\mathbf{f}_\tau\in\mathbf{L}^{2+\delta_1}(\Omega)$ *for some* $\delta_1>0$ *and* $\mathbf{J}_{0\tau}\in\mathbf{L}^q(\Omega)$ *with* q *given as in (14) then* $\mathbf{u}_\tau\in\mathbf{W}^{1,2+\delta}(\Omega)$ *for some* $\delta>0$*. Furthermore* $(\mathbf{u}_\tau,\mathbf{H}_\tau,T_\tau)$ *is unique under small data.*

Proof. See the proof of Theorems 2.1 and 2.3.

Introducing the forms as

(F1) $\alpha_0(\tau,\mathbf{u},\mathbf{v})=\nu\int_\Omega\zeta(\tau)(\varrho(\tau)D\mathbf{u}):(\varrho(\tau)D\mathbf{v})dx=\nu\int_\Omega A(\tau):(D\mathbf{u}D\mathbf{v})dx$

(F2) $\alpha_1(\tau,\mathbf{u},\mathbf{v})=\int_\Omega\zeta(\tau)(\varrho(\tau)\nabla\mathbf{u}):(\mathbf{v}\otimes\mathbf{u})dx=\int_\Omega B(\tau)\nabla\mathbf{u}:(\mathbf{v}\otimes\mathbf{u})dx$

(F3) $\alpha_2(\tau,\mathbf{H},\mathbf{v})=\mu\int_\Omega\zeta(\tau)\Big(((\varrho(\tau)\nabla)\times\mathbf{H})\times\mathbf{H}\Big)\cdot\mathbf{v}dx$

(F4) $\alpha_3(\tau,\mathbf{f},T,\mathbf{v})=\int_\Omega\zeta(\tau)\Big(\mathbf{f}-\mathbf{G}(T)T\Big)\cdot\mathbf{v}dx$

(F5) $\beta_1(\tau,\mathbf{H},\mathbf{w})=\int_\Omega((\varrho(\tau)\nabla)\times\mathbf{H})\cdot((\varrho(\tau)\nabla)\times\mathbf{w})\;dx$

(F6) $\beta_2(\tau,\mathbf{u},\mathbf{H},\mathbf{w})=\sigma\mu\int_\Omega\zeta(\tau)(\mathbf{u}\times\mathbf{H})\cdot((\varrho(\tau)\nabla)\times\mathbf{w})\;dx$

(F7) $\beta_3(\tau,\mathbf{J}_0,\mathbf{w})=\int_\Omega\zeta(\tau)\mathbf{J}_0\cdot((\varrho(\tau)\nabla)\times\mathbf{w})\;dx$

(F8) $\gamma_1(\tau,T,\eta)=k\int_\Omega A(\tau):(\nabla T\otimes\nabla\eta)dx$

(F9) $\gamma_2(\tau,\mathbf{u},T,\eta)=\int_\Omega\zeta(\tau)\mathbf{u}\cdot(\varrho(\tau)\nabla)T\eta dx=\int_\Omega B(\tau):(\mathbf{u}\otimes\nabla T)\eta dx$

(F10) $\gamma_3(\tau,T,\eta)=\alpha\int_{\Gamma_N}T\eta\omega(\tau)ds$

(F11) $\gamma_4(\tau,f,\eta)=\int_\Omega f\eta\zeta(\tau)dx$

(F12) $\gamma_5(\tau,h,\eta)=\int_{\Gamma_N}h\eta\omega(\tau)ds$

the following corollary can be stated.

Corollary 5.9. *Assume (H5). Let* $|\tau|\le\tau_1$ *and* τ_1 *be small enough, then there exist realvalued functions* g_i *satisfying* $g_i(\tau)=o(\tau)$*,* $i=0,...,11$ *and forms* $\tilde\alpha_i(\tau,...),i=0,1,2,3,$ $\tilde\beta(\tau,...)$*,* $i=1,2,3,$ *and* $\tilde\gamma(\tau,...)$*,* $i=1,\cdots,5$*, such that the following statements are valid.*

(B1) *For all* $\mathbf{u},\mathbf{v}\in\mathbf{V}$

$$\begin{aligned}\alpha_0(\tau,\mathbf{u},\mathbf{v}) \;&=\; \alpha_0(0,\mathbf{u},\mathbf{v})+\tau\alpha_{0,\tau}(0,\mathbf{u},\mathbf{v})+\widetilde{\alpha_0}(\tau,\mathbf{u},\mathbf{v})\\ &\alpha_{0,\tau}(0,\mathbf{u},\mathbf{v})=\nu\int_\Omega A'(0):(D\mathbf{u}D\mathbf{v})dx\\ &\widetilde{\alpha_0}(\tau,\mathbf{u},\mathbf{v})\le g_0(\tau)\|\mathbf{u}\|_1\|\mathbf{v}\|_1.\end{aligned}$$

(B2) *For all* $\mathbf{u}, \mathbf{v} \in \mathbf{V}$

$$\begin{aligned} \alpha_1(\tau, \mathbf{u}, \mathbf{v}) &= \alpha_1(0, \mathbf{u}, \mathbf{v}) + \tau\alpha_{1,\tau}(0, \mathbf{u}, \mathbf{v}) + \widetilde{\alpha_1}(\tau, \mathbf{u}, \mathbf{v}) \\ &\qquad \alpha_{1,\tau}(0, \mathbf{u}, \mathbf{v}) = \int_\Omega B'(0)\nabla\mathbf{u} : (\mathbf{v} \otimes \mathbf{u})dx \\ &\qquad \widetilde{\alpha_1}(\tau, \mathbf{u}, \mathbf{v}) \le g_1(\tau)\|\mathbf{u}\|_1^2\|\mathbf{v}\|_1. \end{aligned}$$

(B3) *For all* $\mathbf{H} \in \mathbf{H}^1(\Omega)$ *and* $\mathbf{v} \in \mathbf{V}$

$$\begin{aligned} \alpha_2(\tau, \mathbf{H}, \mathbf{v}) &= \alpha_2(0, \mathbf{H}, \mathbf{v}) + \tau\alpha_{2,\tau}(0, \mathbf{H}, \mathbf{v}) + \widetilde{\alpha_2}(\tau, \mathbf{H}, \mathbf{v}) \\ &\qquad \alpha_{2,\tau}(0, \mathbf{H}, \mathbf{v}) = \mu\int_\Omega ((\varrho'(0)\nabla) \times \mathbf{H}) \times \mathbf{H} \cdot \mathbf{v}dx \\ &\qquad \widetilde{\alpha_2}(\tau, \mathbf{H}, \mathbf{v}) \le g_2(\tau)\|\nabla \times \mathbf{H}\|\|\mathbf{H}\|_1\|\mathbf{v}\|_1. \end{aligned}$$

(B4) *For all* $\mathbf{f} \in \mathbf{L}^2(\Omega)$*,* $T \in Z$ *and* $\mathbf{v} \in \mathbf{V}$

$$\begin{aligned} \alpha_3(\tau, \mathbf{f}, T, \mathbf{v}) &= \alpha_3(0, \mathbf{f}, T, \mathbf{v}) + \widetilde{\alpha_3}(\tau, \mathbf{f}, T, \mathbf{v}) \\ &\qquad \widetilde{\alpha_3}(\tau, \mathbf{f}, T, \mathbf{v}) \le g_3(\tau)\big(\|\mathbf{f}\| + G^{\#}\|T\|\big)\|\mathbf{v}\|. \end{aligned}$$

(B5) *For all* $\mathbf{H}, \mathbf{w} \in \mathbf{V}(rot)$

$$\begin{aligned} \beta_1(\tau, \mathbf{H}, \mathbf{w}) &= \beta_1(0, \mathbf{H}, \mathbf{w}) + \tau\beta_{1,\tau}(0, \mathbf{H}, \mathbf{w}) + \widetilde{\beta_1}(\tau, \mathbf{H}, \mathbf{w}) \\ &\qquad \beta_{1,\tau}(0, \mathbf{H}, \mathbf{w}) = \int_\Omega A'(0) : (\nabla \times \mathbf{H}) \otimes (\nabla \times \mathbf{w})dx \\ &\qquad \widetilde{\beta_1}(\tau, \mathbf{H}, \mathbf{w}) \le g_4(\tau)\|\mathbf{H}\|_1\|\mathbf{w}\|_1. \end{aligned}$$

(B6) *For all* $\mathbf{u} \in \mathbf{V}$*,* $\mathbf{H} \in \mathbf{V}(rot)$*,* $\mathbf{w} \in \mathbf{V}(rot)$

$$\begin{aligned} \beta_2(\tau, \mathbf{u}, \mathbf{H}, \mathbf{w}) &= \beta_2(0, \mathbf{u}, \mathbf{H}, \mathbf{w}) + \tau\beta_{2,\tau}(0, \mathbf{u}, \mathbf{H}, \mathbf{w}) + \widetilde{\beta_2}(\tau, \mathbf{u}, \mathbf{H}, \mathbf{w}) \\ &\qquad \beta_{2,\tau}(0, \mathbf{u}, \mathbf{H}, \mathbf{w}) = \sigma\mu\int_\Omega (\mathbf{u} \times \mathbf{H}) \cdot ((\varrho'(0)\nabla) \times \mathbf{w})dx \\ &\qquad \widetilde{\beta_2}(\tau, \mathbf{u}, \mathbf{H}, \mathbf{w}) \le g_5(\tau)\|\mathbf{u} \times \mathbf{H}\|_2\|\mathbf{w}\|_{\mathbf{V}(rot)}. \end{aligned}$$

(B7) *For all* $\mathbf{J}_0 \in \mathbf{L}^2(\Omega)$*,* $\mathbf{w} \in \mathbf{V}(rot)$

$$\begin{aligned} \beta_3(\tau, \mathbf{J}_0, \mathbf{w}) = \beta_2(0, \mathbf{J}_0, \mathbf{w}) + \tau\beta_{3,\tau}(0, \mathbf{J}_0, \mathbf{w}) + \widetilde{\beta_3}(\tau, \mathbf{J}_0, \mathbf{w}) \\ \beta_{3,\tau}(0, \mathbf{J}_0, \mathbf{w}) = \int_\Omega \mathbf{J}_0 \cdot ((\varrho'(0)\nabla) \times \mathbf{w})dx \\ \widetilde{\beta_3}(\tau, \mathbf{J}_0, \mathbf{w}) \le g_6(\tau)\|\mathbf{J}_0\|\|\nabla \times \mathbf{w}\|. \end{aligned}$$

(B8) *For all* $T, \eta \in Z$

$$\begin{aligned}\gamma_1(\tau, T, \eta) &= \gamma_1(0, T, \eta) + \tau\gamma_{1,\tau}(0, T, \eta) + \widetilde{\gamma_1}(\tau, T, \eta)\\ &\quad \gamma_{1,\tau}(0, T, \eta) = k\int_\Omega A'(0) : (\nabla T \otimes \nabla\eta)dx\\ &\quad \widetilde{\gamma_1}(\tau, T, \eta) \leq g_7(\tau)\|T\|_1\|\eta\|_1.\end{aligned}$$

(B9) *For all* $\mathbf{u} \in \mathbf{V}$ *and* $T, \eta \in Z$

$$\begin{aligned}\gamma_2(\tau, \mathbf{u}, T, \eta) &= \gamma_2(0, \mathbf{u}, T, \eta) + \tau\gamma_{2,\tau}(0, \mathbf{u}, T, \eta) + \widetilde{\gamma_2}(\tau, \mathbf{u}, T, \eta)\\ &\quad \gamma_{2,\tau}(0, \mathbf{u}, T, \eta) = \int_\Omega B'(0) : (\mathbf{u} \otimes \nabla T)\eta dx\\ &\quad \widetilde{\gamma_2}(\tau, \mathbf{u}, T, \eta) \leq g_8(\tau)\|\mathbf{u}\|_1\|T\|_1\|\eta\|_1.\end{aligned}$$

(B10) *For all* $T \in Z, \eta \in Z$

$$\begin{aligned}\gamma_3(\tau, T, \eta) &= \gamma_3(0, T, \eta) + \tau\gamma_{3,\tau}(0, T, \eta) + \widetilde{\gamma_3}(\tau, T, \eta)\\ &\quad \gamma_{3,\tau}(0, T, \eta) = \alpha\int_{\Gamma_N} T\eta\omega'(0)ds\\ &\quad \widetilde{\gamma_3}(\tau, T, \eta) \leq g_9(\tau)\|T\|_1\|\eta\|_1.\end{aligned}$$

(B11) *For all* $f \in L^2(\Omega)$ *and* $\eta \in Z$

$$\begin{aligned}\gamma_4(\tau, f, \eta) &= \gamma_4(0, f, \eta) + \widetilde{\gamma_4}(\tau, f, \eta)\\ &\quad \widetilde{\gamma_4}(\tau, f, \eta) \leq g_{10}(\tau)\|f\|\|\eta\|.\end{aligned}$$

(B12) *For all* $h \in L^2(\Gamma_N)$ *and* $\eta \in Z$

$$\begin{aligned}\gamma_5(\tau, h, \eta) &= \gamma_5(0, h, \eta) + \tau\gamma_{5,\tau}(0, h, \eta) + \widetilde{\gamma_5}(\tau, h, \eta)\\ &\quad \gamma_{5,\tau}(0, h, \eta) = \int_{\Gamma_N} h\eta\omega'(0)ds\\ &\quad \widetilde{\gamma_5}(\tau, h, \eta) \leq g_{11}(\tau)\|h\|_{\Gamma_N}\|\eta\|_1.\end{aligned}$$

Proof. The expressions (B1)-(B12) are consequence of the following derivatives with respect to τ at $\tau = 0$

$$\zeta'(0) = \text{ div } V(0) = 0, \quad \varrho'(0) = B'(0) = -{}^*JV(0), \quad A'(0) = -2D(V(0)),$$

observing that as in Section 1, $D(V(0))$ denotes the symmetrized part of $JV(0)$, i.e. $D(V(0)) = \frac{1}{2}(JV(0) + {}^*JV(0))$. For the proof see Sokolowski-Zolésio [27, Section 2.13].

Applying Taylor polynomials of degree one we can prove the stability results.

Proposition 5.10. *Under the assumptions of Theorem 5.8, if* $(\mathbf{u}_\tau, \mathbf{H}_\tau, T_\tau)$ *is the transported solution correspondent to* $(\mathbf{u}, \mathbf{H}, T)$ *and the following assumptions are fulfilled*

(M1) $\|\mathbf{f}_\tau - \mathbf{f}\| \leq C|\tau|$

(M2) $\|\mathbf{G}(T_\tau)T_\tau - \mathbf{G}(T)T\|_{6/5} \leq C|\tau|$

(M3) $\|\mathbf{J}_{0\tau} - \mathbf{J}_0\| \leq C|\tau|$

(M4) $\|h_\tau - h\|_{\Gamma_N} \leq C|\tau|$

(M5) $\|f_\tau - f\| \leq C|\tau|$

then we have

$$\|\mathbf{u}_\tau - \mathbf{u}\|_1 \leq C|\tau|; \tag{41}$$
$$\|\mathbf{H}_\tau - \mathbf{H}\|_1 \leq C|\tau|; \tag{42}$$
$$\|T_\tau - T\|_1 \leq C|\tau|, \tag{43}$$

with C denoting different constants.

Proof. For τ small enough and $\xi_i \in [0, \tau]$, $i = 0, ..., 4$, we can write

$$\begin{aligned} \zeta(\tau) &= 1 + \tau\zeta'(\xi_0), \\ \varrho(\tau) &= I + \tau\varrho'(\xi_1), \\ A(\tau) &= I + \tau A'(\xi_2), \\ B(\tau) &= I + \tau B'(\xi_3), \\ \omega(\tau) &= 1 + \tau\omega'(\xi_4), \end{aligned}$$

where $\zeta(\tau) \geq c_{\tau_1} > 0$ for $|\tau| \leq \tau_1$ and A, B, ϱ are positive definite for $|\tau| \leq \tau_1$.

Observing that α_0 is linear with respect to the second argument, we write

$$\begin{gathered} \alpha_0(\tau, \mathbf{u}_\tau, \mathbf{v}) - \alpha_0(0, \mathbf{u}, \mathbf{v}) = \alpha_0(0, \mathbf{u}_\tau - \mathbf{u}, \mathbf{v}) + \\ + \tau\nu \int_\Omega A'(\xi_2) : (D\mathbf{u}_\tau D\mathbf{v}) dx. \end{gathered} \tag{44}$$

Observe that α_i, $(i=1,2,3)$, is no more linear with respect to the secondary argument, thus we write

$$\begin{aligned}\alpha_1(\tau,\mathbf{u}_\tau,\mathbf{v})-\alpha_1(0,\mathbf{u},\mathbf{v})=\alpha_1(0,\mathbf{u}_\tau,\mathbf{v})-\alpha_1(0,\mathbf{u},\mathbf{v})+\\+\tau\int_\Omega B'(\xi_3)\nabla\mathbf{u}_\tau:(\mathbf{v}\otimes\mathbf{u}_\tau)dx;\end{aligned}\tag{45}$$

$$\begin{aligned}&\alpha_2(\tau,\mathbf{H}_\tau,\mathbf{v})-\alpha_2(0,\mathbf{H},\mathbf{v})=\alpha_2(0,\mathbf{H}_\tau,\mathbf{v})-\alpha_2(0,\mathbf{H},\mathbf{v})+\\&+\tau\mu\int_\Omega\Big(\zeta'(\xi_0)(\nabla\times\mathbf{H}_\tau)\times\mathbf{H}_\tau+((\varrho'(\xi_1)\nabla)\times\mathbf{H}_\tau)\times\mathbf{H}_\tau\Big)\cdot\mathbf{v}dx;\end{aligned}\tag{46}$$

$$\begin{aligned}\alpha_3(\tau,\mathbf{f}_\tau,T_\tau,\mathbf{v})-\alpha_3(0,\mathbf{f},T,\mathbf{v})=\alpha_3(0,\mathbf{f}_\tau,T_\tau,\mathbf{v})-\\-\alpha_3(0,\mathbf{f},T,\mathbf{v})+\tau\int_\Omega\zeta'(\xi_0)\Big(\mathbf{f}_\tau-\mathbf{G}(T_\tau)T_\tau\Big)\cdot\mathbf{v}dx.\end{aligned}\tag{47}$$

Considering that $\mathbf{u}$ is the particular case $(\tau=0)$ to the perturbed $\mathbf{u}_\tau$ it follows that

$$\text{RHS of (44)}+\text{RHS of (45)}=\text{RHS of (46)}+\text{RHS of (47)}.$$

If we set $\mathbf{v}=\mathbf{u}_\tau-\mathbf{u}$ and arguing as in the proof of Theorem 2.3 we get

$$\begin{aligned}\Big(\frac{\nu}{2}-C_2^2\|\mathbf{u}\|_1\Big)\|\mathbf{u}_\tau-\mathbf{u}\|_1^2\le\frac{C}{\nu}\Big(\mu\|(\nabla\times\mathbf{H}_\tau)\times\mathbf{H}_\tau-(\nabla\times\mathbf{H})\times\mathbf{H}\|_{6/5}\\+\|\mathbf{f}_\tau-\mathbf{f}\|+\|\mathbf{G}(T_\tau)T_\tau-\mathbf{G}(T)T\|_{6/5}+C|\tau|\Big)^2.\end{aligned}\tag{48}$$

Now, observing that β_1 and β_3 are linear with respect to the second argument we write

$$\begin{aligned}\beta_1(\tau,\mathbf{H}_\tau,\mathbf{w})-\beta_1(0,\mathbf{H},\mathbf{w})&=\beta_1(0,\mathbf{H}_\tau-\mathbf{H},\mathbf{w})+\\&+\tau\int_\Omega A'(\xi_2):(\nabla\times\mathbf{H}_\tau\otimes\nabla\times\mathbf{w})dx;\\\beta_3(\tau,\mathbf{J}_{0\tau},\mathbf{w})-\beta_3(0,\mathbf{J}_0,\mathbf{w})&=\beta_3(0,\mathbf{J}_{0\tau}-\mathbf{J}_0,\mathbf{w})+\\&+\tau\int_\Omega B'(\xi_3):(\mathbf{J}_{0\tau}\otimes\nabla\times\mathbf{w})dx,\end{aligned}$$

while the remaining term reads

$$\begin{aligned}\beta_2(\tau,\mathbf{u}_\tau,\mathbf{H}_\tau,\mathbf{w})-\beta_2(0,\mathbf{u},\mathbf{H},\mathbf{w})=\beta_2(0,\mathbf{u}_\tau,\mathbf{H}_\tau,\mathbf{w})-\\-\beta_2(0,\mathbf{u},\mathbf{H},\mathbf{w})+\tau\sigma\mu\int_\Omega B'(\xi_3):\Big((\mathbf{u}_\tau\times\mathbf{H}_\tau)\otimes\nabla\times\mathbf{w}\Big)dx.\end{aligned}$$

Considering that $\mathbf{H}$ is the particular case ($\tau = 0$) to the perturbed $\mathbf{H}_\tau$, we set $\mathbf{w} = \mathbf{H}_\tau - \mathbf{H}$ to get the following estimate

$$\|\mathbf{H}_\tau - \mathbf{H}\|_1 \leq \|\mathbf{J}_{0\tau} - \mathbf{J}_0\| + \sigma\mu\|\mathbf{u}_\tau \times \mathbf{H}_\tau - \mathbf{u} \times \mathbf{H}\| + C|\tau|. \tag{49}$$

Next

(N1) $\gamma_1(\tau, T_\tau, \eta) - \gamma_1(0, T, \eta) = \gamma_1(0, T_\tau - T, \eta) + \tau k \int_\Omega A'(\xi_2) : (\nabla T_\tau \otimes \nabla \eta) dx$;

(N2) $\gamma_2(\tau, \mathbf{u}_\tau, T_\tau, \eta) - \gamma_2(0, \mathbf{u}, T, \eta) = \gamma_2(0, \mathbf{u}_\tau, T_\tau, \eta) - \gamma_2(0, \mathbf{u}, T, \eta) + \tau \int_\Omega B'(\xi_3) : (\mathbf{u}_\tau \otimes \nabla T_\tau) \eta dx$;

(N3) $\gamma_3(\tau, T_\tau, \eta) - \gamma_3(0, T, \eta) = \gamma_3(0, T_\tau - T, \eta) + \tau\alpha \int_{\Gamma_N} T_\tau \eta \omega'(\xi_4) ds$;

(N4) $\gamma_4(\tau, f_\tau, \eta) - \gamma_4(0, f, \eta) = \gamma_4(0, f_\tau - f, \eta) + \tau \int_\Omega f_\tau \eta \zeta'(\xi_0) dx$;

(N5) $\gamma_5(\tau, h_\tau, \eta) - \gamma_5(0, h, \eta) = \gamma_5(0, h_\tau - h, \eta) + \tau \int_{\Gamma_N} h_\tau \eta \omega'(\xi_4) ds$.

We set $\eta = T_\tau - T$ to get

$$\frac{k}{2}\|\nabla(T_\tau - T)\|^2 + \alpha\|T_\tau - T\|^2_{\Gamma_N} \leq \frac{C}{k}\Big(\|(\mathbf{u}_\tau - \mathbf{u}) \cdot \nabla T\|_{6/5} + \\ + \|f_\tau - f\| + \|h_\tau - h\|_{\Gamma_N} + C|\tau|\Big)^2. \tag{50}$$

Now we add the three inequalities (48),(49),(50) and from assumptions (M1) -(M5), we get (41)-(43).

Finally, we are in the position to formulate the existence theorem for the material derivative of our problem.

Definition 5.11. The following limit in the function space norm $\mathcal{H}$

$$\dot{f} = \lim_{\tau \to 0} \frac{f(\tau) - f(0)}{\tau}$$

is called the strong material derivative $\dot{f}$ of f in $\mathcal{H}$.

Remark 5.12. The shape derivative u' of $u(\tau)$ in the direction of the vector field V is defined by the formula $u' = \dot{u} - \nabla u \cdot V$ provided that there exists the material derivative $\dot{u}$.

We recall that $A(0) = B(0) = \varrho(0) = I$, $\zeta(0) = \omega(0) = 1$, $\dot{\zeta} = \zeta'(0) = 0$, $\dot{\varrho} = \varrho'(0)$, $\dot{A} = A'(0)$, $\dot{B} = B'(0)$ and $\dot{\omega} = \omega'(0)$, and we state the following result on the existence of material derivatives.

Theorem 5.13. *Assume (H2), (H5),* $\dot{\mathbf{f}} \in \mathbf{L}^2(\Omega), \dot{\mathbf{J}} \in \mathbf{L}^2(\Omega), \dot{f} \in L^2(\Omega), \dot{h} \in L^2(\Gamma_N)$ *and moreover (10) holds with constants* a *and* b *given as*

$$\begin{aligned} a &= \frac{\nu}{\mu\sigma}\|\dot{\mathbf{J}}_0\| \\ b &= \frac{\nu}{\mu\sigma} - \left(\|\dot{\mathbf{f}}\| + \frac{\dot{G}^{\#}}{k}(\|\dot{f}\| + \|\dot{h}\|_{\Gamma_N}) \right), \end{aligned}$$

then the triple $(\dot{\mathbf{u}}, \dot{\mathbf{H}}, \dot{T}) \in \mathbf{V} \times \mathbf{V}(rot) \times Z$ *satisfies*

$$\nu \int_\Omega (\dot{A}D\mathbf{u} + D\dot{\mathbf{u}}) : D\mathbf{v}dx + \int_\Omega (\dot{B}\nabla\mathbf{u} + \nabla\dot{\mathbf{u}}) : (\mathbf{v} \otimes \mathbf{u})dx + \int_\Omega \nabla\mathbf{u} : (\mathbf{v} \otimes \dot{\mathbf{u}})dx$$

$$= \mu \int_\Omega \Big(((\dot{\varrho}\nabla) \times \mathbf{H}) \times \mathbf{H} + (\nabla \times \dot{\mathbf{H}}) \times \mathbf{H} + (\nabla \times \mathbf{H}) \times \dot{\mathbf{H}} \Big) \cdot \mathbf{v}dx +$$

$$+ \int_\Omega \Big(\dot{\mathbf{f}} - \dot{\mathbf{G}}(T)T - \mathbf{G}(T)\dot{T} \Big) \cdot \mathbf{v}dx, \quad \forall \mathbf{v} \in \mathbf{V};$$

$$\int_\Omega ((\dot{\varrho}\nabla) \times \mathbf{H} + \nabla \times \dot{\mathbf{H}}) \cdot (\nabla \times \mathbf{w})\ dx + \int_\Omega (\nabla \times \mathbf{H}) \cdot ((\dot{\varrho}\nabla) \times \mathbf{w})\ dx =$$

$$= \sigma\mu \int_\Omega \Big(\dot{\mathbf{u}} \times \mathbf{H} + \mathbf{u} \times \dot{\mathbf{H}} \Big) \cdot (\nabla \times \mathbf{w})dx + \sigma\mu \int_\Omega (\mathbf{u} \times \mathbf{H}) \cdot ((\dot{\varrho}\nabla) \times \mathbf{w})dx +$$

$$+ \int_\Omega \dot{\mathbf{J}}_0 \cdot (\nabla \times \mathbf{w})dx + \int_\Omega \mathbf{J}_0 \cdot (\dot{\varrho}\nabla) \times \mathbf{w}dx, \quad \forall \mathbf{w} \in \mathbf{V}(rot);$$

$$\begin{aligned} &k \int_\Omega (\dot{A}\nabla T + \nabla\dot{T}) \cdot \nabla\eta dx + \int_\Omega (\dot{B} : \mathbf{u} \otimes \nabla T + \dot{\mathbf{u}} \cdot \nabla T + \mathbf{u} \cdot \nabla\dot{T})\eta dx \\ &+\alpha \int_{\Gamma_N} (\dot{T} + T\dot{\omega})\eta ds = \int_\Omega \dot{f}\eta dx + \int_{\Gamma_N} (\dot{h} + h\dot{\omega})\eta ds, \quad \forall \eta \in Z; \end{aligned}$$

and the following estimates

$$\begin{aligned} \|\dot{T}\|_1 &\leq C\Big((1 + \|\mathbf{u}\|_1 + \|\dot{\mathbf{u}}\|_1)\|T\|_1 + \|\dot{f}\| + \|\dot{h}\|_{\Gamma_N} + \|h\|_{\Gamma_N} \Big); \\ \|\dot{\mathbf{u}}\|_1 &\leq C\Big((\|\dot{\mathbf{H}}\|_1 + \|\mathbf{H}\|_1)\|\mathbf{H}\|_1 + \|\dot{\mathbf{f}}\| + \dot{G}^{\#}\|T\|_1 + G^{\#}\|\dot{T}\|_1 + \|\mathbf{u}\|_1 \Big); \\ \|\dot{\mathbf{H}}\|_1 &\leq C\Big(\|\mathbf{H}\|_1 + \mu\sigma(\|\dot{\mathbf{u}} \times \mathbf{H}\| + \|\mathbf{u} \times \dot{\mathbf{H}}\| + \|\mathbf{u} \times \mathbf{H}\|) + \|\dot{\mathbf{J}}_0\| + \|\mathbf{J}_0\| \Big). \end{aligned}$$

Proof. We subtract the perturbated solution and the transported solution and we pass to the limit with τ tending to 0 (for details see [7] for analogous proof).

6 Concluding remarks

In order to overcome the problem of loosing divergence free behavior we can apply Piola transform which is given by the following mapping:

$$P_I : \mathbf{V} \to \mathbf{V}^\tau;$$

$$\mathbf{v} \mapsto (J\mathcal{T}_\tau \cdot \mathbf{v}) \circ \mathcal{T}_\tau^{-1}.$$

Denoting

$$\hat{\mathbf{u}}_\tau := (J\mathcal{T}_\tau)^{-1} \cdot (\mathbf{u}_\tau \circ \mathcal{T}_\tau) \text{ defined on } \Omega$$

and

$$\mathbf{u}_\tau = P_I(\hat{\mathbf{u}}_\tau) \text{ is defined on } \Omega_\tau,$$

the mapping P_I can be applied on velocity field and also on magnetic field to conserve the divergenceless and that $\mathbf{u} \cdot \mathbf{n} = 0$ and $\mathbf{H} \cdot \mathbf{n} = 0$. By the same method as in Section 5 we get the stability and material derivative for $\hat{\mathbf{u}}$ and then we just apply the inverse mapping to conclude the results in [7].

Remark 6.1. In [7] we get the stability depending not only on the data but also on assumption of behavior of $\mathbf{H}$, but it is not the case in our present problem.

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